

BF gravity and the Immirzi parameter

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Abstract

We propose a novel BF-type formulation of real four-dimensional gravity, which generalizes previous models. In particular, it allows for an arbitrary Immirzi parameter. We also construct the analogue of the Urbantke metric for this model.

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Real general relativity can be formulated as a constrained first-order BF-type theory of the form [1] (for an earlier alternative approach, see [2])

$$S[B, A, \phi, \mu] = \int [B^{IJ} \wedge F_{IJ}(A) + G(B, \phi, \mu)] , \quad (1)$$

where $B^{IJ} = -B^{JI}$ are six real 2-forms, $A^{IJ} = -A^{JI}$ is an $SO(3,1)$ connection, with $F^{IJ} = dA^{IJ} + A^I_K \wedge A^{KJ}$ its curvature (Lorentz indices $I, J, \dots = 0, 1, 2, 3$ are raised and lowered with the Minkowski metric η_{IJ}). $G(B, \phi, \mu)$ denotes a constraint quadratic in the 2-forms B^{IJ} . Its role is to implement that, for some tetrad e^I , the 2-forms take the form $B^{IJ} = *(e^I \wedge e^J)$, with $*$ the duality operator on Lorentz indices ($*^2 = -1$). When the constraint is solved, substitution back in the action of this specific form would then recover general relativity in its first-order tetrad formulation. For Euclidean gravity, we have $\eta^{IJ} \rightarrow \delta^{IJ}$, the connection is valued in $SO(4)$, and $*^2 = +1$.

The constraint is of the form

$$G(B, \phi, \mu) = -\frac{1}{2}\phi_{IJKL}B^{IJ} \wedge B^{KL} + \mu H(\phi) , \quad (2)$$

with ϕ_{IJKL} a Lagrange multiplier with obvious symmetries $\phi_{IJKL} = -\phi_{JIKL} = -\phi_{IJLK} = \phi_{KLIJ}$. It has 21 independent components. Since this is one too many, as the B^{IJ} have

36 and the e^I have 16, one needs to impose some extra conditions on ϕ , via the 4-form Lagrange multiplier μ , which sets $H(\phi) = 0$. The following conditions were considered [1, 3, 4]:

$$H_1 = \phi_{IJ}{}^{IJ} = 0, \quad (3)$$

$$H_2 = \phi_{IJKL}\varepsilon^{IJKL} = 0. \quad (4)$$

The solution of the associated constraints on the B^{IJ} in terms of an arbitrary real tetrad e^I leads to a modified version of tetrad gravity,

$$S[e, A] = \alpha \int *(e^I \wedge e^J) \wedge F_{IJ}(A) + \beta \int e^I \wedge e^J \wedge F_{IJ}(A). \quad (5)$$

where for H_1 , we have the four independent solutions $\alpha = \pm 1$ and $\beta = \pm 1$, whereas from H_2 one obtains the four independent solutions $\alpha = \pm 1, \beta = 0$, or $\alpha = 0, \beta = \pm 1$. Apart from the annoying sign ambiguities, the first term gives the Einstein-Hilbert action, and, at least for non-degenerate tetrads, the second term vanishes on shell.

These formulations have been put to use both in the Lorentzian and the Euclidean cases in the context of various real four-dimensional approaches to quantum gravity that have come to be known as spin foam models [5], or Feynman diagrams for gravity [6]. These approaches were motivated initially by canonical quantum gravity, first in its complex version based on the Ashtekar phase space variables [7], then in its real version based on the Ashtekar-Barbero phase space variables [8]. In the latter there is an arbitrary real parameter, which enters in the spectra of geometric operators, which has come to be known as the Immirzi parameter [9]. It corresponds to α/β in (5). One would expect it to play a role also in these various four-dimensional approaches, but as their continuum analog is given by the BF models described above, this appears not to be the case.

The purpose of this note is to point out that the Immirzi parameter emerges naturally if one considers a condition more general than (3) and (4). This seems to have been overlooked in previous investigations. (This possibility was advocated first in [10], but at the time its relevance was not recognized.) We assume that, rather than vanishing separately, the invariants are proportional so that

$$H_3 = a_1 \phi_{IJ}{}^{IJ} + a_2 \phi_{IJKL}\varepsilon^{IJKL} = 0, \quad (6)$$

with a_1, a_2 arbitrary constants. The previous cases are obtained as either one vanishes. We emphasize that this possibility is available only in four dimensions. In the BF formulation of higher dimensional gravity, there is no difference between (3) and the equivalent of (4) [11].

Variation of the action with respect to the Lagrange multipliers, and taking the appropriate traces, gives that the constraints on the B^{IJ} are

$$B^{IJ} \wedge B^{KL} = \frac{1}{6}(B^{MN} \wedge B_{MN})\eta^{[I|K|}\eta^{J]L} + \frac{\epsilon}{12}(B^{MN} \wedge *B_{MN})\varepsilon^{IJKL}, \quad (7)$$

$$2a_2 B^{IJ} \wedge B_{IJ} = \epsilon a_1 B^{IJ} \wedge *B_{IJ}, \quad (8)$$

with $\epsilon = 1$ in the Euclidean case and $\epsilon = -1$ in the Lorentzian one. It is easy to show that in this case the 2-forms are given in terms of some tetrad e^I by

$$B^{IJ} = \alpha * (e^I \wedge e^J) + \beta e^I \wedge e^J, \quad (9)$$

with α, β arbitrary non-vanishing real parameters that satisfy $\alpha^2 \neq \beta^2$ (the case $\alpha^2 = \beta^2$ gives the special case (3)). It is only a matter of calculation to prove that this condition is necessary. For the sufficiency it is convenient to break explicit internal Lorentz invariance and express the constraints in terms of B^{0i}, B^{ij} ($i, j = 1, 2, 3$), and the problem reduces to a set of constraints on three 2-forms, say B^{0i} , with (9) as its immediate solution. The ratio β/α is determined algebraically by the ratio $a_2/a_1 = (\alpha^2 + \epsilon\beta^2)/4\alpha\beta$, as follows from (8). Substitution of (9) in (1) gives the tetrad action (5), with arbitrary Immirzi parameter α/β .

When either a_1 or a_2 vanishes, we see by inspection of (8) that we obtain the degenerate cases in which $B^{IJ} \wedge B_{IJ}$ or $B^{IJ} \wedge *B_{IJ}$ vanish respectively. This modifies (7), and the form of the B^{IJ} is restricted accordingly.

It is interesting to construct the metric directly in terms of the B^{IJ} . Guided by its analogue in the self-dual case, which has come to be known as the Urbantke metric [12, 13], we consider the two possible invariants, of weight two, cubic in the B^{IJ} ,

$$\mathcal{G}^{\mu\nu} = \tilde{B}^{\mu\alpha IJ} B_{\alpha\beta}{}^{KL} \tilde{B}^{\beta\nu MN} \eta_{IN} \epsilon_{JKLM}, \quad (10)$$

$$\mathcal{H}^{\mu\nu} = \tilde{B}^{\mu\alpha IJ} B_{\alpha\beta}{}^{KL} \tilde{B}^{\beta\nu MN} \eta_{IN} \eta_{JK} \eta_{LM}, \quad (11)$$

where $\tilde{B}^{\mu\nu IJ} = \varepsilon^{\mu\nu\rho\sigma} B_{\rho\sigma}{}^{IJ}$, $\varepsilon^{\mu\nu\rho\sigma}$ denotes the spacetime Levi-Civita tensor density, and greek indices denote spacetime indices.

Using (9) the following is obtained

$$\mathcal{G}^{\mu\nu} = 6\alpha (\epsilon\alpha^2 + 3\beta^2) g^{\mu\nu}, \quad (12)$$

$$\mathcal{H}^{\mu\nu} = 3\beta (\epsilon\beta^2 + 3\alpha^2) g^{\mu\nu} \quad (13)$$

where $g^{\mu\nu} = e^\mu_I e^\nu_J \eta^{IJ}$, g its determinant and e^μ_I the inverse tetrad. These expressions allow us to express the (densitized) metric in various ways in terms of the B^{IJ} . In the special cases of H_1 and H_2 is enough one of the metrics $\mathcal{G}^{\mu\nu}$ or $\mathcal{H}^{\mu\nu}$. Considering that we are assuming $\alpha^2 \neq \beta^2$, the natural choice is

$$gg^{\mu\nu} = \frac{\epsilon}{3(\beta^4 - \alpha^4)} \left(\beta \mathcal{H}^{\mu\nu} - \frac{1}{2} \alpha \mathcal{G}^{\mu\nu} \right). \quad (14)$$

The Hamiltonian formulation of the action (5) has been performed in [14] for arbitrary parameters, and leads to the Ashtekar-Barbero phase space variables [15] (see also [16]). Alternatively, one can perform the canonical analysis directly from the BF action (1), with the condition (6), arriving at the same result [10].

We emphasize that the constraint (2) is what distinguishes a topological field theory (the BF part), from a theory with local degrees of freedom like gravity. It is crucial to understand

how to impose it at the quantum level. In particular, our observation suggests that in current discrete models which use a triangulation of spacetime, the partition function should be extended to include non-simple representations associated with the triangles, and change appropriately the intertwiners. One expects that under this generalization, one would arrive at the same conclusion of the canonical approaches: a one-parameter ambiguity in the spectra of geometric operators such as area and volume.

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